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OPTIMAL STIFFENING OF HOLES UNDER EQUIBIAXIAL TENSION

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Abstract—In this paper, the optimal configuration—in a remote uniform tension field—is investigated for stiffening rings (made of a different material) in a perforated elastic plate. The complex variable approach and the Kolosov–Muskhelishvili potentials are used to determine the unknown shape of the multiconnected region in which we pose the equations of equilibrium. It turns out that, for any number and relative spacing of the holes, the ring boundaries should represent equal-strength contours. This finding is extended to the case of multilayered and composite materials with elastic moduli varying continuously in a given direction.

1. INTRODUCTION

Stiffening rings are widely used for reinforcing hole boundaries in perforated thin elastic plates. Made of a different elastic material, they permit optimization of the stress and strain fields with the aid of certain criteria, most commonly those of equivalency and minimum energy. The first criterion is of local nature; apparently, originally advanced by Mansfield (1950), it calls for coincidence of the stress tensors at all corresponding points of the plate at hand and of a solid plate under the same loads, specified by the values P, Q at infinity parallel to the X, Y-axes (Fig. 1). In other words, the perturbed fields induced by the holes and the rings (not counting the inner stresses in the latter) are expected to be mutually counteractive. The second criterion suggested by Mikhailovskii and Shaunin (1978) requires that sum of two integrals, representing the potential energy of deformation of the plates and the rings, respectively, will be minimal. The chosen criterion is commonly met by determining suitable configuration for the holes and rings, possibly subjected to supplementary isoperimetric restrictions on their area. Except for the case of a single hole under symmetrical ($P = Q$) load, where a concentric cicular ring of particular width is the obvious solution, the procedure of elastic field conjugation along the interface results in significant non-linearity. This procedure has been studied qualitatively by Aleksandrov and Kurshin (1966) and the configuration of a stiffened single hole was found by Kurshin and Rastorguev (1980) numerically using expansion in terms of the small parameter $\varepsilon = (P-Q)/(P+Q)$, under the simplifying assumption of zero bending stiffness for the energy-wise optimal ring.

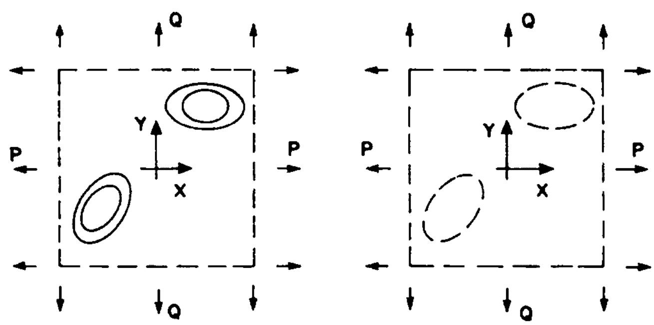

Fig. 1. The equivalent stiffening criterion. The stress fields in the plate with stiffened holes (a) and in the uncut plate (b) are the same in their mutual region.

In the present paper, the following open problem is explicitly solved: optimize the reinforcement simultaneously for both criteria with $\varepsilon = 0$. The optimal configuration turns out to be the so-called equal-strength contours extensively studied in the past [see review by Banichuk (1983)] in the context of the closely-related optimization problem for an unstiffened perforated plate.

While no numerical studies in that problem are available, it appears worthwhile to compare our analytical results with future finite element studies.

2. GOVERNING EQUATIONS

Consider a plate with m arbitrary arranged non-intersecting holes, located in the plane of the complex variable $z = x+iy$. These holes (Fig. 2) are stiffened by annular rings made of a different material, of variable width with outer (L_k) and inner (l_k) edges, $k = 1, 2, \ldots, m$, whose sets $L = \cup L_k$, $l = \cup l_k$ decompose the plane into an outer domain S_0, occupied by the plate itself, a middle domain $S + \cup S_k$, containing the rings and finally an inner domain $S_- = \cup S_k$ inside l.

The stress field of such a plate is described (Muskhelishvili, 1975) by $(m+1)$ pairs of functions $\varphi_j(z)$, $\psi_j(z)$ holomorphic respectively in the domains S_j, $j = 0, 1, \ldots, m$. At infinity these functions satisfy the asymptotic requirements

$$\varphi_0(z) = a_0 z + 0(|z|^{-1}), \quad \psi_0(z) = b_0 z + 0(|z|^{-1}),$$
$$4a_0 = P+Q, \quad 2b_0 = Q-P, \quad |z| \to \infty, \tag{1}$$

the load conditions

$$\varphi_k(\xi) + \xi\overline{\varphi_k'(\xi)} + \overline{\psi_k(\xi)} = C_k, \quad \xi \in l_k, \quad k = 1, 2, \ldots, m, \tag{2}$$

and are linked to L through the continuous contact conditions:

$$\varphi_0(\eta) + \eta\overline{\varphi_0'(\eta)} + \overline{\psi_0(\eta)} = \varphi_k(\eta) + \eta\overline{\varphi_k'(\eta)} + \overline{\psi_k(\eta)} + D_k \tag{3}$$

$$\mu_1[\kappa_0\varphi_0(\eta) - \eta\overline{\varphi_0'(\eta)} - \overline{\psi_0(\eta)}] = \mu_0[\kappa_1\varphi_k(\eta) - \eta\overline{\varphi_k'(\eta)} - \overline{\psi_k(\eta)}],$$
$$\eta \in L_k, \quad k = 1, 2, \ldots, m, \tag{4}$$

where C_k, D_k are arbitrary constants, $\kappa_i = (3-v_i)/(1+v_i)$, and v_i, μ_i, $i = 0, 1$ are the elastic moduli of the plate and rings respectively.

Each optimum criterion imposes particular supplementary requirements on the potentials $\varphi_j(z)$, $\psi_j(z)$. Thus, with the minimum-energy case they read (Kurshin and Rastorguev, 1980; Vigdergauz, 1989):

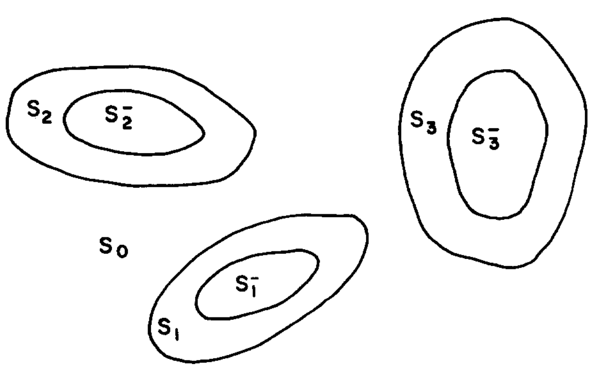

Fig. 2. z-plane decomposition.

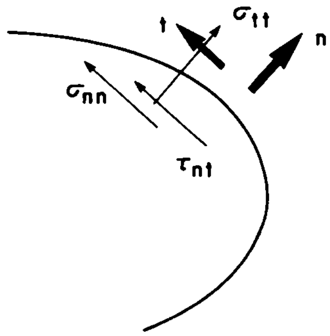

Fig. 3. The local coordinate system and stress tensor components on the optimal boundary.

$$w^{(k)}(\xi) = d_k, \quad \xi \in l_k, \tag{5}$$

$$w^{(0)}(\eta) - w^{(k)}(\eta) = \beta_k, \quad \eta \in L_k, \quad k = 1, 2, \ldots, m, \tag{6}$$

where $2w^{(j)}(z) = [I_1^2(z)/2(1+v_i) + I_2(z)]\mu_i^{-1}$ is the strain energy density, expressed in terms of the stress invariants

$$I_1(z) = \sigma_{xx}(z) + \sigma_{yy}(z) = 4 \operatorname{Re} \varphi_j'(z), \tag{7}$$

$$I_2(z) = \tau_{xy}^2(z) - \sigma_{xx}(z)\sigma_{yy}(z) = |\bar{z}\varphi_j''(z) + \psi_j(z)|^2 - [\operatorname{Re} \varphi_j'(z)]^2;$$
$$z \in S_j, \quad j = 0, 1, \ldots, m. \tag{8}$$

The constants of α_k, β_k are Lagrange multipliers for the given areas (F, f) of the domains (S_-, S) in the variational derivation of (5), (6).

With the equivalency case, the following identities should be satisfied:

$$\varphi_0(z) = a_0 z, \quad \psi_0(z) = b_0 z, \quad z \in S_0. \tag{9}$$

These identities correspond to the homogeneous stress field of a solid plate under identical loads. In this case only the total area $(F+f)$ is given and the area f of the ring is determined together with its configuration.

Conditions (5) and (6) or (9) are redundant within the framework of the boundary-value problem (1)–(4), and therefore may be used for determining the configuration L, l. If the holes are unstiffened, i.e. the plate occupies the domain $\Sigma = S + S_0$ and the edges l_k and functions $\varphi_k(z)$, $\psi_k(z)$ are irrelevant—then the optimum solution is represented by equal-strength curves along which the following identity is satisfied:

$$\sigma_{tt}(\zeta) = \text{const.}, \quad \zeta \in \Gamma, \quad \Gamma = \cup \Gamma_k, \tag{10}$$

where $(\sigma_{nn}, \sigma_{tt}, \tau_{nt})$ are the components of the stress tensor in the local coordinate system (n, t) along the normal and tangential vectors to Γ_k (Fig. 3). By the boundary conditions $\sigma_{nn}(\zeta)$, $\tau_{nt}(\zeta) = 0$. Thus $\operatorname{Re} \varphi'(\zeta) = I_1(\zeta)/4$ is constant on Γ, as it follows from (9) and (7). So we obtain, taking the asymptotics (1) into account, that the function $\varphi'(z)$ is constant everywhere in Σ, i.e. $\varphi(z) = a_0 z$ for all z. Hence (10) is equivalent to the first optimality condition (9) as well as to (5), while (2) is transformed as follows $(\chi(z) \equiv 2\psi_0(z) - 2b_0)$:

$$\chi(\zeta) = -a_0\bar{\zeta} - b_0\zeta - C_k, \quad (\zeta \in \Gamma_k), \quad |\chi(z)| \to 0, \quad (|z| \to \infty). \tag{11}$$

The equal-strength contours themselves are found analytically (Cherepanov, 1974) or numerically (Vigdergauz, 1976), via the inverse boundary-value problem, by conformal mapping from a certain standard plane of variable θ onto the z-plane.

As we know (Courant, 1950) a standard θ-plane with m parallel slits can be mapped onto any m-connected domain of the complex z-plane with a point at infinity. When $m > 2$, the mapping $w_0(\theta)$ which has the form $w_0(\theta) = C\theta + w(\theta)$, where $w(\theta)$ is bounded at infinity, depends on a set of $3m$ real parameters, six of which (e.g. one end point and a length of a certain slit, one fixed point on this slit and a center point of another slit) can be specified arbitrarily and C is a scaling factor. For $m = 2$ the slits may be located on the X-axis and the number of parameters is equal to 3. Consequently, a system of equal-strength contours, if it exists, forms a multi-parameter family. The limit of variation of the parameters can be found from geometrical considerations. It should be emphasized that these contours really exist for any number and spacing of the holes, as determined by this set of parameters specifying (up to the factor C) the conformally equivalent classes of domains Σ. Within each of the classes the solution is unique. In the presumption of cyclical or biaxial domain symmetry the number of parameters does not depend on the number of holes and may be reduced to one (that particular case will be denoted by λ). As a result, the mapping function $w_0(\theta)$ can be obtained in the form of elliptic integrals (Cherepanov, 1974). Figure 4 shows a quadrant of a system of two equal-strength holes on the X-axis as a function of λ— the length of two equal slits with fixed center coordinates in the standard θ-plane.

This arbitrary choiced of the parameters permits the solution of the optimal reinforcement problem (1)–(6), (9) under symmetric loads ($b_0 = 0$, $\chi(z) = 2\psi_0(z)$). We shall prove that in this case the boundaries of the optimal rings are sets of equal-strength curves, corresponding to different values of the parameters: $L = \Gamma(\lambda_0)$, $l = \Gamma(\lambda_1)$, the invariant $I_1(z)$ being constant in each homogeneous part of the domain $\Sigma(\lambda)$, and interaction of the materials along the contours L_k reduces to uniform normal pressure with no shear stresses. In the sequel, whenever required, the relevant values are referred to as functions of λ.

Indeed, differentiation of the identity (11) with respect to ξ along Γ, with the aid of relation $|d\xi/d\zeta| = 1$ shows that on the boundary of its domain of definition, represented by $\lambda = \lambda_1$, the function $\chi'(z)$ has a uniform modulus:

$$|\chi'(\zeta,\lambda_1)| = a_0, \quad (\zeta \in \Gamma(\lambda_1)). \tag{12}$$

We next find another set Λ of uniform modulus curves for the prescribed value $a_0 < a_1$ (this inequality follows from the maximum principle of harmonic functions). For the function $a_0 a_1^{-1}\chi'(z,\lambda_1)$ on Λ relation (12) will obviously meet, hence Λ necessarily coincides with a certain equal-strength boundary for another value λ falling completely within $\Sigma(\lambda_0)$: $\Lambda = \Gamma(\lambda_0)$. The corresponding function $\chi(z,\lambda_0)$, satisfies the identity

$$\chi'(\zeta,\lambda_0) = a_0 a_1^{-1}(\lambda_0)\chi'(\zeta,\lambda_1), \quad \zeta \in \Gamma(\lambda_0), \tag{13}$$

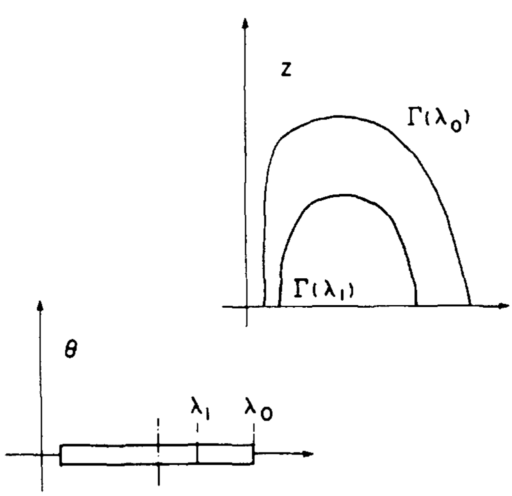

Fig. 4. The scheme of conformal mapping from standard 0-plane to z-plane. The parameter λ governing the form of optimal curves is shown.

and is easily integrated along $\Gamma(\lambda_0)$. Integration constants must be equal on both sides, as by Harnack's theorem a function holomorphic in $\Sigma(\lambda_0)$ and vanishing at infinity cannot take nonzero values on the domain's boundary. Thus, functions $\chi(z,\lambda_0)$, $a_0a_1^{-1}(\lambda_0)\chi(z,\lambda_1)$ themselves equal each other along $\Gamma(\lambda)$ and consequently throughout L (their common domain of definition).

To find the relationship between a_1 and λ_0 we consider $\delta(\lambda)$—the residual at infinity of the function $\chi(z,\lambda)$ for any λ. Recalling (11) we have:

$$2\pi\delta(\lambda) = \int_{\Gamma(\lambda)} \chi(\zeta,\lambda)\,d\zeta = -a_0\int_{\Gamma(\lambda)} \bar{\zeta}\,d\zeta.$$

The last integral is directly proportional (Muskhelishvili, 1975) to the area $s(\lambda)$ of the region $S_-(\lambda)$ inside $\Gamma(\lambda)$. Thus, the following identity is obtained

$$a_0F = a_1(F+f), \quad (F = s(\lambda_1), \quad f = s(\lambda_0)-s(\lambda_1)). \tag{14}$$

The properties of function $\chi(z,\lambda_1) \equiv 2\psi_0(z,\lambda_1)$ are sufficient for solving the problem of optimal reinforcement. Let

$$\begin{aligned} \varphi_0(z) &= a_0z, \quad \psi_0(z) = B_0\psi_0(z,\lambda_0), \quad z\in S_0,\\ \varphi_k(z) &= A_1z, \quad \psi_k(z) = B_1\psi_0(z,\lambda_1), \quad z\in S_k. \end{aligned} \tag{15}$$

Where non-essential constant terms have been omitted on the right-hand side of each identity in (15). Constants A_1, B_0, B_1 have still to be determined. From (15) we find:

$$\psi_k(\xi) = B_1\left[\frac{a_0}{2}\bar{\xi}+C_k(\lambda_1)\right], \quad \xi\in l_k, \quad k = 1,2,\ldots,m,$$

$$\psi_0(\eta) = B_0\left[\frac{a_0}{2}\bar{\eta}+C_k(\lambda_0)\right], \quad \psi_k(\eta) = \frac{B_1F}{F+f}\left[\frac{a_0}{2}\bar{\eta}+C_k(\lambda_0)\right], \quad \eta\in L_k. \tag{16}$$

Substitution of (16) in (2)–(4), with the coefficients of ξ and η equated, results in a set of linear algebraic equations:

$$\begin{aligned} &4A_1+a_0B_1 = 0, \quad 4A_1+a_0(1-\Delta_0f)B_1-a_0B_0 = a_0,\\ &4\mu_0(\kappa_1-1)A_1-2\mu_0a_0(1-\Delta_0f)B_1+2\mu_1a_0B_0 = \mu_1(\kappa_0-1)a_0, \end{aligned} \tag{17}$$

with the solution

$$A_1 = \frac{a_0}{4}R_0\mu_1(\kappa_0+1), \quad B_1 = -\frac{R_0}{2}\mu_1(\kappa_0+1),$$

$$B_0 = (1-\Delta_0f)R_0\mu_1(\kappa_0+1)-1, \quad R_0^{-1} = \mu_0(\kappa_1+1)+2\Delta_0f(\mu_1-\mu_0), \tag{18}$$

here $\Delta_0f = f/(F+f)$ denotes the ratio of total area of the rings to that of the holes.

It is readily verified that the necessary conditions of energy-wise optimality are met by the potentials (5), (6) and also that the following relations are valid:

$$\begin{aligned} &I_1(z) = 4a_0(z\in S), \quad I_1(z) = 4a_1, \quad (z\in S_1),\\ &2\sigma_{nn}(\eta) = \mu_1B_0\Delta_0f(\kappa_0+1)a_0, \quad (\eta\in L),\\ &\sigma_{tt}^{(0)}(\eta) = a_0-\sigma_{nn}(\eta), \quad \sigma_{tt}^{(1)}(\eta) = a_0R_0\mu_1(\kappa_0+1)\left(1-\frac{\Delta_0f}{2}\right),\\ &\sigma_{tt}(\xi) = a_0R_0\mu_1(\kappa_0+1), \quad (\xi\in l_0). \end{aligned} \tag{19}$$

It should be noted that it is natural to minimize the strain energy only after elimination of the divergence due to the infinite extent of the corresponding integral. To this end only the finite part of S with a sufficiently remote boundary is taken into account by Mikhailovskii *et al.* (1978) and Kurshin *et al.* (1980). While outside that boundary the stress field is assumed to be homogeneous with uniform energy density proportional to a_0^2, which, for $b_0 = 0$, corresponds to a solid plate. Alternatively, this part may be omitted from the energy integrand while retaining the initial infinite region, in which convergence is provided by (1). For the problem in hand, both approaches yield the same optimality conditions (5)–(16) [see Kurshin and Rastorguev (1980) and Vigdergauz (1989)]. Hence they are equivalent. With the second approach we obtain the desired results, making use of expressions (7), (8), (14) and identities (15), (18). The perturbation of total energy is:

$$2U = \sum_{k=1}^{n} \int_{S_k} w^{(k)}(z)\,\mathrm{d}z + \int_{S_0} \left[w^{(0)}(z) - \frac{(1-v_0)a_0^2}{16\mu_0(1+v_0)} \right] \mathrm{d}z$$

$$= \frac{1}{8\mu_1(1+v_1)} \int_F^{(F+f)} [(1-v_1)A_1^2 + 2(1+v_1)a_0^2 B_1^2 F^2] \frac{\mathrm{d}s}{s^2}$$

$$+ \frac{1}{4\mu_1} \int_{(F+f)}^{\infty} B_1^2 a_0^2 (F+f) \frac{\mathrm{d}s}{s^2}$$

$$= \mu_1^{-1} \left[\frac{1-v_1}{8(1+v_1)} + (1-\Delta_0 f) f \frac{a_0^2 R_0 \mu_1^2 (\kappa_0+1)^2}{16} + \frac{a_0^2 (F+f)[\Delta_0 f R \mu_1 (\kappa_0+1) - 1]}{4\mu_0} \right]. \quad (20)$$

For given elastic moduli μ_j, v_j, $j = 0, 1$ and areas F, f this value of U is the smallest possible within the framework of the adopted formulation of the example problem, which involves the curvilinear boundaries of the stiffening rings as a set of admissible functions, the energy of deformation as an objective functional and the given values F, f as restrictions.

It is of special interest that the problem solution is immediately obtained by an equi-strength concept rather than by a multi-step procedure of numerical optimization.

Relations (19), (20) generalize the results of Vigdergauz (1988) in terms of optimization of the stress field in a plate with solid ($\Delta_0 f \equiv 1$) non-identical inclusions for $b_0 = 0$.

The relative area of the equivalent reinforcement, determined from (18) via the condition $\psi_0(z) \equiv 0$, is

$$\Delta_0 f = R_0[\kappa_0(\mu_1+1) - \kappa_1(\mu_0+1)].$$

Inasmuch as $0 \leqslant \Delta_0 f \leqslant 1$, the preceding equality imposes the restriction

$$\mu_0(\kappa_1+1) \leqslant \mu_1(\kappa_0+1) \quad (21)$$

under which equivalent reinforcement actually exists. The area of reinforcing material decreases as its relative rigidity increases.

It should be emphasized, that the proposed relations are completely exact, so they can be regarded as attainable lower bounds for the same functions associated with stiffening rings of any other shape. Therefore, the gain from applying optimal contours can be estimated by means of the quantity $\rho = U_m/U$ where U_m is the energy for the non-optimal rings being compared with the smallest value U from (20).

Thus, the numerical solution of the direct problem (1)–(4) for two identical circular holes and concentric stiffening rings of particular relative width ($\Delta_0 f = 0.25$) especially obtained by the author on the basis of series expansion for the potentials (1) and (2), gives $\rho = 1.22$, when $\mu_1/\mu_0 = 1$, $\kappa_1/\kappa_0 = 0.75$, $h_1 = 1$, $d_1 = 0.4$.

Here h_1 is the distance between the centers of the circles, d_1 is their diameter, serving at the same time as conformal mapping parameter for the corresponding double connected optimal region. The ratio ρ rapidly increases as parameter λ tends to its limit value, which is obviously equal to one.

3. MULTILAYERED STIFFENING

A similar approach is valid for stiffening rings of N distinct layers. In keeping track with the previously obtained results, the stresses are optimal in terms of (5), (6) if the boundaries of all layers are as before sets of equi-strength curves for the parameter λ_q, $q = 0, 1, \ldots, N$. As has been proved in Section 2, the governing parameters of the problem are identical in that case for all holes, accordingly, we change the notation and use the subscript q for labeling regions of homogeneity beginning with the plate itself ($q = 0$).

Again as before, the potentials are sought in the form:

$$\varphi_q(z) = A_q z, \quad \psi_q(z) = B_q \psi_0(z, \lambda_q), \quad z \in S_q, \quad q = 0, 1, \ldots, N. \tag{22}$$

Here $4A_0 = a_0$, while all other constants are determined by substituting expressions (22) in the readily transformed contact conditions (3), (4), for the pair of adjoining materials; viz.

$$\mathbf{v}_q = R_q M^{(q)} \mathbf{v}_{q-1}, \quad q = 1, 2, \ldots, N, \tag{23}$$

and in the equilibrium condition at the load-free boundary of the Nth layer:

$$4A_N + a_0 B_N = 0. \tag{24}$$

Here,

$$R_q^{-1} = 2a_0 \mu_{q-1}(1 - \Delta_{q-1} f)(\kappa_q + 1), \quad \Delta_{q-1} f = 1 - s(\lambda_q)/s(\lambda_{q-1}).$$

Vector $\mathbf{v}_q = (A_q, B_q)$, and $M^{(q)}$ is a (2×2) matrix with elements:

$$M_{11}^{(q)} = 2a_0(1 - \Delta_{q-1} f)[2\mu_{q-1} + \mu_q(\kappa_q - 1)],$$
$$M_{12}^{(q)} = 2a_0^2(1 - \Delta_{q-1} f)(\mu_{q-1} - \mu_q),$$
$$M_{21}^{(q)} = 8[\mu_{q-1}(\kappa_{q-1} - 1) - \mu_q(\kappa_q - 1)],$$
$$M_{22}^{(q)} = 2a_0[\mu_{q-1}(\kappa_q - 1) + 2\mu_q].$$

The set of $(2N+1)$ linear algebraic equations (23) is of the band type, and admits an explicit solution by a two-step method.

In the first step the vector $\mathbf{v}_N$ is expressed in terms of $v_0 = (\frac{1}{4}a_0, b_0)$ through recursive application of identity (23):

$$\mathbf{v}_N = \prod_{q=1}^{N} M^{(q)} \mathbf{v}_0 \tag{25}$$

and subsequently determined by a further substitution of (25) in (24). In the second step all vectors $\mathbf{v}_q$ follow consequently.

The structure of the extra condition of equivalence associated with $b_0 = 0$ is obvious and resembles (21). It is omitted here for the sake of brevity.

Letting the number of layers tend to infinity, the finite-difference equations are transformed into differential form, with the discrete subscript q replaced by the continuous parameter λ within the interval $G \in [\lambda_0, \lambda_N]$. This operation corresponds to the model of a thin-layered composite [see Bolotin and Novichkov (1980)], whose elastic properties vary in a single direction along the curves transverse to the family of equi-strength contours $\Gamma(\lambda)$, $\lambda \in G$.

Within this procedure, the following differential equation serves as an analog to (23):

$$-\frac{\partial \mathbf{v}(\lambda)}{\partial \lambda} = R(\lambda)M(\lambda)\mathbf{v}_0. \tag{26}$$

Here,

$$R^{-1}(\lambda) = \mu(\lambda)[\kappa(x)-1]$$

and the elements of matrix $M(\lambda)$ have the form:

$$M_{11}(\lambda) = [\mu'(\lambda)(\kappa(\lambda)-1)-2\mu(\lambda)\kappa'(\lambda)],$$

$$M_{12}(\lambda) = -a_0\frac{\mu'(\lambda)}{2\mu(\lambda)},$$

$$M_{21}(\lambda) = 2a_0^{-1}[\mu(\lambda)\kappa'(\lambda)-\mu'(\lambda)\kappa(\lambda)][1-s'(\lambda)/s(\lambda)],$$

$$M_{22}(\lambda) = 2[1-s'(\lambda)/s(\lambda)]\mu(\lambda)\kappa^{-1}(\lambda).$$

The solution of eqn (26) possesses the following form, on account of boundary condition $v(\lambda_0) = v_0$,

$$\mathbf{v}(\lambda) = \exp[M(\lambda_0)-M(\lambda)]\mathbf{v}_0, \tag{27}$$

vector $\mathbf{v}_0$ is then determined from (24). In the case of piecewise constant moduli, relationship (27) is transformed into the corresponding expressions for multilayered reinforcements.

In concluding, it is worth noting that Kurshin and Rastorguev (1980) derived two necessary conditions for energy-wise optimization of hole reinforcement by means of momentless elastic filament with variable cross-section instead of a stiffening ring. The first condition, which requires uniform deformation of the desired contour, can be shown to be equivalent to the equal strength condition. The second condition is essentially a non-linear ordinary differential equation, which ties the shear stresses τ_{nt}, along the contact line, with the line's curvature and with filament rigidity. Regarded as unknown functions of the contour arc, they can be found only numerically and with significant effort. In our case, where $\tau_{nt} = 0$, a complete two-dimensional consideration of the elastic properties of the reinforcement has greatly simplified the problem at hand, preserving its linearity and permitting an explicit solution for multi-connected regions with certain degrees of symmetry. For more general cases the multi-parametrical equations of equal-strength contours possess very complicated forms, so the proposed method has no advantage over FEM and other effective numerical methods.

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